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RADIAL VELOCITIES

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The measurement of the velocities of astronomical objects (or speeding motorists) depends on the Doppler-Fizeau principle first enunciated for acoustic waves in 1842 by Christian Johann Doppler (1803–1853). Its application to optical waves was proposed in 1848 by the French physicist Armand Hippolyte Louis Fizeau (1819–1896) following his early measures of the velocity of light, using the fact that spectrum lines define particular frequencies or wavelengths. The modern formulation of the principle based on the restricted theory of relativity of Albert Einstein is state in the form:

$$\nu = \nu_0 \frac{(1 - v^2/c^2)^{1/2}}{1 + (v/c)\cos(\theta/c)},$$

where ν_0 is the frequency emitted by a certain source and ν that measured by an observer relative to whom the source is moving with velocity v at an angle θ to the line of sight. The velocity of light c, is 299,793 km s^{-1}. If $\theta = 0$ this becomes

$$\nu = \nu_0 \frac{(1 - v/c)^{1/2}}{(1 + v/c)^{1/2}}, \tag{1}$$

while if $\theta = 90°$

$$\nu = \nu_0 (1 - v^2/c^2)^{1/2} \tag{2}$$

the latter describing the transverse Doppler effect which so far has received no application in astronomy.

Equation (1) can be written as

$$\lambda = \lambda_0 \frac{(1 + v/c)^{1/2}}{(1 - v/c)^{1/2}}, \tag{3}$$

where λ_0 is the wavelength of the emitted radiation. For velocities small with respect to c equation (3) becomes

$$\lambda = \lambda_0 (1 + v/2c)(1 + v/2c) \sim \lambda_0 (1 + v/c),$$

or if $\Delta\lambda = \lambda - \lambda_0$, $\Delta\lambda/\lambda_0 = v/c$, where v is a (positive) velocity of recession corresponding to a displacement of the observed spectrum line towards longer (redder) wavelengths. It is in this form that the relation is applied in most stellar studies where recession velocities rarely exceed $100\,\mathrm{km\,s^{-1}}$ and never more than about $500\,\mathrm{km\,s^{-1}}$.

In extragalactic and quasar studies equation (3) is usually written as

$$z = (\lambda - \lambda_0)/\lambda_0 = (1 + v/c)^{1/2}(1 - v/c)^{-1/2} - 1.$$

If $v/c = 0.9$, $z = 3.36$ which is the approxomate limit of observed values at the present time. It has been remarked that occasionally writers fail to subtract the unity and write $z = \lambda/\lambda_0$ producing results too large by one.

VELOCITY CORRECTIONS

A measurement of a radial velocity contains an element due to the rotation of the earth, an element due to the motion of the earth round the Sun, and an element due to the motion of the Sun with respect to the mean of the nearby stars. These elements, depending on the context and purpose for which the measurement has been made, may be removed by computing a variety of corrections. In the case of observations of external galaxies, quasars or some other objects it may be desired to apply corrections for the motion of the solar neighborhood around the center of the Galaxy or for the motion of our own Galaxy with respect to an average motion of various clusters or groups of galaxies. (Eberhard, 1933).

A. *Correction for rotation of the Earth.* The equatorial rotational velocity of the Earth is $0.465\,\mathrm{km\,s^{-1}}$ so that the value for an observatory at latitude φ is $0.465\cos\varphi\,\mathrm{km\,s^{-1}}$ which, when projected on the line of sight to a star at declination δ, hour angle H west, yields a component $0.465\cos\varphi\cos\delta\sin H$ which must be *subtracted* from the observed velocity of recession. Depending on the accuracy likely to be legitimately obtained or desired, the simple expression $0.465\cos\varphi$ which neglects the ellipticity of the Earth's figure should be replaced by $0.465\rho\cos\varphi$ where ρ is the radius vector from the Earth's centre to the observatory taking the equatorial radius to be unity. In the majority of cases the introduction of this feature is unwarranted.

B. *Correction for the Earth's orbital motion.* The mean velocity of the Earth in its orbit is $29.78\,\mathrm{km\,s^{-1}}$ and is denoted by V. Let λ, β be the mean ecliptic longitude and latitude of the star observed and O the longitude of the Sun. Let Π be the longitude of the Sun at perigee and e the eccentricity of the Earth's orbit ($= 0.016722$). It can be shown that the correction to be *added* to a velocity of recession is $V\cos\beta\sin(O - \lambda) + Ve\cos\beta\sin(\Pi - \lambda)$. The second term is always less than $0.5\,\mathrm{km\,s^{-1}}$ but should be included for a great majority of determinations of stellar radial velocity especially of later type stars with many narrow lines. In the past a great many relatively simplified approximate methods of computation have been developed, such as, for example using the rate of change of the equatorial coordinates of the Sun (X, Y, Z) and the equatorial coordinates of the object (α, δ) but with modern methods of computation, even a pocket calculator can readily handle the formula given above in which

$$\cos\beta = \cos\varepsilon\sin\delta - \sin\varepsilon\sin\alpha\sin\delta$$
$$\cos\lambda\cos\beta = \cos\alpha\cos\delta,$$

where ε is the obliquity of the ecliptic and α, δ are the equatorial coordinates of the object observed.

C. The correction to be applied to remove the effect of the motion of the Earth about the mass center of the Earth-Moon system, neglecting inclination, ellipticity and eccentricity of the Moon's orbit is $-0.0124\cos\beta\sin(\Lambda - \lambda)\,\mathrm{km\,s^{-1}}$ where Λ is the Moon's longitude. In all but the most sophisticated observations this can be neglected.

D. The motion of the Sun with respect to the nearest stars is at $19.7\,\mathrm{km\,s^{-1}}$ in the direction $\alpha = 271°$, $\delta = +30°$ (1900). The so-called basic solar motion, representing deviations from circular motion in the Galaxy is usually quoted as $+9\,\mathrm{km\,s^{-1}}$ towards the Galactic center, $+12\,\mathrm{km\,s^{-1}}$ in the plane and $+7\,\mathrm{km\,s^{-1}}$ towards the north Galactic pole. The velocity of all the neighboring stars in circular motion about the Galactic center is estimated as $250\,\mathrm{km\,s^{-1}}$ perpendicular to the center in a clockwise direction as viewed from the north side. These and other larger scale motions are deductions from a great many radial velocity studies and do not represent ordinary reductions. They will only be encountered as causing corrections to be applied to large scale studies after such of the corrections A, B, C have been applied to the raw data.

EARLY HISTORY

Sir William Huggins (1824–1910) a wealthy London amateur was the first to apply the Doppler-Fizeau principle to the measurement of the radial velocity of a star. This was to Sirius, observed visually, in 1868 which was found to have a recession velocity of 29 miles per second, clearly uncorrected for the Earth's motion. In the years immediately following velocities for a number of first magnitude stars were determined visually. These operations were extremely hard to carry out and involved a devastating strain on the eyesight of the observes. William Wallace Campbell (1862–1938) one of the great pioneers of the subject is said to have ruined his sight in this way.

Photographic methods were introduced about a century ago and remained dominant for about 90 years especially with the introduction of telescopes and spectrographs designed for optimum speed and resolution, the establishment of observatories designed for astrophysical studies on sites selected for best atmospheric conditions and a steady improvement in the speed and range of spectral sensitivity of photographic emulsions.

The measurement of radial velocities is a technique applied in many different forms to practically all areas of astrophysics. The following is a list of some of the principal problems studied during the photographic period: — determination of velocities of stars in general, especially the brighter ones: concentration on the behavior of star clusters, and of special types of stars such as early type (O and B) stars and Cepheid variables for studies of Galactic rotation; studies of particular types of stars both constant and variable for the determination of kinematic parallaxes and kinematics of the Galaxy; rotational studies of the sun, planets and some stars; discovery and study of spectroscopic binary stars; studies of emission objects in the Galaxy, such as planetary nebulae; initial studies of extragalactic nebulae leading to discovery of the general expansion of the universe; studies of rotation of some galaxies leading to estimates of masses.

In the early days most of this work was done with prism spectrographs of relatively low dispersion (10–30 A/mm at $H\gamma$) which gave a much lower dispersion towards the red end of the spectrum. Towards the end of the interwar period grating spectrographs became more widely used and a beginning was made with the introduction of Schmidt cameras as camera elements which improved the overall speed of the equipment.

If we take a survey of the situation at about 1930 we find radial velocity work being done at such observatories as Victoria, British Columbia; Mount Wilson, California; the Royal Observatory at the Cape, even though its large refractor was not particularly suitable for this work; at Ottawa, Michigan, and Yerkes and at Lick Observatory, California and at its southern D. O. Mills expedition to Santiago, Chile. J. H. Moore (1932) published his general catalogue of radial velocities of stars, nebulae and clusters containing results for 6739 stars, counting visual doubles as two stars. Moore adopted final values for 6354 stars. He also listed velocities for 133 gaseous nebulae, for 18 globular clusters and for 90 extragalactic nebulae. There was almost complete coverage for stars down to magnitude 5.5, many of sixth magnitude and a minority fainter than seventh.

TECHNIQUES AND ACCURACY

The basic technique used was to obtain a star spectrum, suitably broadened by trailing the image on the spectrograph slit during the exposure, flanked above and below by a comparison spectrum locally generated by passing its light through the ends of the slit with the central part, where the star spectrum had been, temporarily blocked off. The comparison spectrum, often that of an iron arc or sometimes of a gas tube such as hydrogen, was supposed to follow the same optical path as that of the star, and to fill the collimator of the spectrograph in the same way. The developed plates were then placed in a micrometer measuring machine with some standard comparison line as near to a standard measure as possible. The relation between the micrometer measure n and wavelength was then taken to have the form

$$\lambda - \lambda_0 = k/(n_0 - n)^\alpha$$

where λ_0, n_0, and α were constants for a particular instrumental arrangement in the prism case. This relation is usually referred to as the Hartmann formula, though in most cases α may be taken to be unity. For a grating dispersion the relation is $\lambda = \lambda_1 + k_1 n$ where λ_1 and k_1, are constants for that set up. In each case the constants can be determined from the values of n determined for certain recognised lines in the comparison spectrum with wavelengths known from physical tables. These formulae are not exact except at the wavelengths chosen to determine the contants which occur in them. However calculated values of n can be compared with measured values of n for intermediate lines in the comparison spectrum and a correction curve plotted to pass from the computed values to the measured values. Values of n are usually measured in millimeters to an accuracy of a micron.

The determination of the radial velocity will depend on the measures of a selection of stellar lines, in the ideal case based on their identification and the adoption of wavelengths based on laboratory values. We compute values of n for the plate being measured using the appropriate formula, to these values we apply corrections deduced by interpolation from the correction curve. This give values of n which would apply if the recession velocity were zero. The measured values $n_{\rm obs}$ will differ from these calculated values $n_{\rm calc}$ because of the existence of the recession velocity, V. For each line we can write

$$f(n_{\rm obs} - n_{\rm calc}) = V$$

where the factor f corresponding, say, to a difference of one micron on the plate will vary with wavelength but may be calculated using the approximate formula. The result for the plate and its error can be computed as the mean for all the lines used and their deviations from the mean. In practice, because it has been found that measures may have a systematic personal tendency to hack to the left or right of an absorption line the plate is measured both in the direction of increasing wavelength and then by moving it on the microscope state, in the reverse direction. The values used in constructing the correction curve and thereafter are the fractional part of $n_{\rm dir} - n_{\rm rev}$. In such a case the difference to be considered is a half micron, and for the prismatic case, $f = 150k/\lambda(n_0 - n)^2 = 150D/\lambda$ where D is the dispersion in Angstroms per millimeter. This increases more rapidly with wavelength than in the grating case. As an example take $\lambda = 4500$, $D = 15$. Then if $V = 100$, $\Delta\lambda = 1.5$, $\Delta n = 0.1$ for the case of direct measurement only, or 200 half microns in the double measurement case. In this case $f = 0.5$ and $f \times 200 = 100\,{\rm km\,s^{-1}}$.

This example is an indication of the measurement accuracy which has to be achieved if velocities good to $1\,{\rm km\,s^{-1}}$ or better are to be attained.

THE CLASSICAL METHODS IN PRACTICE

In the days when it was usual for observatories to have official programs in which many staff collaborated rather than separate research projects by individual staff members, many observatories undertook general radial velocity programs essentially using such methods. Among the most prominent, especially those cited in Moore's catalog were Victoria, B.C., Lick Observatory including the D. O. Mills expedition to Chile, Mount Wilson, Yerkes, Ottawa, Allegheny, the Royal Observatory at the Cape of Good Hope and Michigan and McDonald. Less important were such places as Simeis, Pulkovo, Perkins, Lowell and others. In 1948 the Radcliffe Observatory at Pretoria became very important, together with work at the Commonwealth Observatory at Canberra and the Argentine National Observatory out-station at Bosque Alegre. In the north Kitt Peak and Haute Provence became active.

A catalog which largely antedates these last named contributions was produced by Ralph Elmer Wilson (1963) and contains results for 15106 objects almost all stellar and including all of Moore's results.

H. A. Abt and E. S. Biggs (1972) produced a bibliography of known stellar radial velocities. This has one disadvantage, namely that for some stars, the same measures are duplicated, but this not a serious matter if one examines the results critically. Evans (1984) accumulated 7823 new records now available from the International Data Center at Strasbourg. In 1922 the International Astronomical Union established Commission No. 30 for the study of radial velocity data. This Commission has sponsored a symposium, No. 30, Batten and Heard (1967), and a colloquium, No. 88 (Davis Philip and Latham, 1985) both of which include extended discussions of the topics sketched in this contribution. The latter contains a bibliography of radial velocity papers 1981–1984. (Davis Philip, 1985).

Both Moore's and Wilson's catalogues devote considerable attention to the comparison of results from different observatories which suggested that for some observatories and/or some spectral types there might be systematic differences of several kilometers per second between measures of the same star. This fact stimulated an intensive study of the sources of differences and a search for ways of removing them and converting all measures to absolute ones. It was clear that there were intrinsic differences in the behavior of different spectrographs, which tended to be eliminated with improvements in design and in technique, with special attention to care in complete filling of the spectrograph collimator, correction of the effects of atmospheric dispersion, investigation of flexure effects and so forth. There remained differences of technique of reduction, selection of star lines, and agreement on radial velocities of standard stars. For high dispersion spectra there was often little doubt about the proper element identification and wavelength assignment to a given star line though for many early type stars there would be few lines to measure and in some cases broadening of the lines made measures made by bisecting them with a hair line in a measuring machine uncertain. For later type stars the spectral features were often blends of several lines especially under lower dispersions and the mean wavelength of a given blended feature might vary with small changes in spectral type or, as became clearer after World War II, with luminosity class. It became clear as results for fainter stars were sought by the adoption of lower spectrographic dispersions that the measurement of a large number of features did not conduce to the reduction of probable errors. At Victoria, especially under the leadership of W. M. Petrie, schemes of adopted wavelengths of particular features appropriate to O and B stars, A-type stars, and solar-type stars were adopted, usually up to some 15–20 in each case, for spectra in various ranges of dispersions (Detailed references in Petrie, 1962). These made the results more consistent internally and allowed corrections to observations of standard stars to be made, though it left open the question whether the results for the different spectral class groups really referred to the same zero point. Many of the stars recommended as standard by the IAU in various reports were chosen because results from several observatories agreed based on independent measures on relatively high dispersion, though with the same perversity which has affected the selection of photometric standard stars which afterwards turned out to be variable, some have been later found to be of variable radial velocity. A good deal of the work on standarisation of O and B type velocities was undertaken by the Dominion Observatory at Victoria, B.C., and the Radcliffe Observatory at Pretoria. A program of determination of velocities of southern standard stars of later types was undertaken by Evans using the Pretoria equipment in 1957 (Evans et al 1957). At this stage velocities of later type standard stars were thought to be accurate to $0.5\,\mathrm{km\,s^{-1}}$ or better. Velocities of O and B type standard stars were good to 1 or $2\,\mathrm{km\,s^{-1}}$ since in many cases their lines were broad and difficult to measure using the then standard technique of bisecting the lines by means of a hair line in the focus of a microscopes as the stage supporting the plate was moved. The fatigue of masurement was to some extent removed by the introduction of projection methods showing the spectra without the monocular vision demanded by a microscope eyepiece (Petrie 1937). Still later, accuracy was improved with the introduction of machines which scanned the profiles of lines photoelectrically and displayed the direct and reverse profiles on cathode ray screens. Measurements were then made by moving the stage until the two profiles were superposed. If the profiles were symmetrical this pushed the accuracy of measurement down to the order of 1 or $2\,\mathrm{km\,s^{-1}}$ even for broad lined program stars from the previous typical range of 4 or $5\,\mathrm{km\,s^{-1}}$. Depending on the dispersion used, typical uncertainties for program stars of later type ranged from about $3\,\mathrm{km\,s^{-1}}$ for low dispersion where in some cases corrections for hour angle had to be applied, to about $1\,\mathrm{km\,s^{-1}}$ for dispersions of the order of 15 Angstroms per millimeter of plate. The problem of absolute standardisation for later type stars could be attacked by observing asteroids for which the radial velocities could be calculated from their ephemerides but no such absolute calibration was available for the earlier type stars and recourse was had to such methods as requiring the velocities of early type stars members of Galactic clusters which also contained later type stars to conform to the mean deduced from the latter. (See Petrie, 1962; Batten, 1985).

OTHER TECHNIQUES

In early recognition of some of these problems especially for low dispersion spectra of late type stars, Johannes Franz Hartmann (1865–1936) proposed the use of spectrum comparators in which the spectrum of a program star could be matched with the spectrum of a standard star whose uncorrected velocity was supposed to be known. The equipment had to be designed specifically for spectra of particular physical dimensions. The spectrum of the standard star was placed on one stage and that of the program star on a moveable one. Images of the two were brought to juxtaposition in the same eyepiece and that of the program star moved by means of a micrometer screw until the comparison lines at a selected portion of the spectrum were in register. The program spectrum was then moved until the stellar features were in register and the displacement required was noted. This was repeated for some half dozen selected regions along the spectrum to give the mean relative displacement of the program star with respect to the standard. For a prism spectrograph of course, the actual physical displacement corresponding to a given velocity decreased from shorter to longer wavelengths. This technique was used at Victoria and for a time at the Cape but one suspects was dropped as a variety of spectrographic dispersions became available because of the inflexibility of the optical arrangements required.

Some 30 years ago P. B. Fellgett (1953) and H. W. Babcock (1955) suggested a modification of this in which a standard spectrum printed positive and a program spectrum of a similar star were to be subjected to a transmitted beam of light which would show a marked reduction indicated photoelectrically when the two were in coincidence so that the gaps in the latter were exactly filled by the blocking of the former. This remained a proposal until R. F. Griffin (1967) took up the idea with the invention of his radial velocity photometer. He replaced the photographic spectrum of the standard star with a ruled mask reproducing a section of the spectrum and sought positions of minimum transmission of light through the mask when the spectrum of a star at the telescope was superposed on it. This scheme has been highly successful and has led among other things to the determination of the orbital parameters of more than sixty spectroscopic binary stars. Griffin's work was initiated at the Cambridge Observatory and similar systems have been installed at a number of other institutions.

During World War II Charles Fehrenbach (Fehrenbach 1947) proposed a sheme for the simultaneous determination of the radial velocities of all the stars in the field of a telescope. This depended on the installation before the objective of a refractor of a specially designed weak compound prism which would disperse the light of each star into a low dispersion spectrum without deviating the mean position of the star on the photographic plate. The observations were to be carried out by making two exposures on the same plate by reversing the prism in its own plane in between. The radial velocities could then be determined by measuring the relative displacements of the forward and reverse spectra brought to juxtaposition by changing the pointing of the telescope slightly in declination between the exposures. The results for a given plate depended on calibration of that plate from known radial velocities of several standard stars which appeared on it. This last condition was particularly hard to achieve and although considerable numbers of results were published, they were of distinctly less precision than would have been the case for more conventional methods and the hope for the production of large numbers of reliable measures was not realised.

With the introduction of new sophisticated detectors such as CCDs which are capable of recording the traces of spectra showing line profiles with signal to

noise ratios of several hundred to one, so that the finest details are significant, new techniques become possible. At high dispersion at a coude focus perhaps only a few tens or even hundreds of Angstroms may be recorded. However if a trace of a standard star is available in the memory of the associated computer, the trace of a program star may be compared with it using a correlation technique to determine the relative displacement and to derive at the telescope the relative radial velocity in certain cases with an accuracy in the range of meters per second. If the same star is observed at different times variability of radial velocity may be detected with great accuracy. In the case of the Sun radial pulsations can be studied. In the case of spectroscopic binaries data to be used in the solution of orbital elements may be obtained.

SOME EARLY ACHIEVEMENTS

The rotation of the Sun was detected in 1871 by H. C. Vogel (1842–1907) who measured the velocity difference between the two limbs. The nature of the rings of Saturn was established by James Edward Keeler (1857–1900) in 1985. He placed a spectrograph slit along the widest part of the rings. The central part of the spectrum showed inclined spectrum lines corresponding to the solid body rotation of the planetary ball and inclined lines from the rings demonstrating Keplerian motion with the innermost parts moving fastest. This verified the conclusion advanced by James Clerk Maxwell (1831–79) in his Adams Prize Essay of 1859 that the rings must be composed of a wealth of discrete solid particles rather than of either liquid or a solid section of a disk.

The first demonstration of a spectroscopic binary shown by variable splitting of the stellar lines at different epochs was made for the case of Mizar reported by Edward Charles Pickering (1846–1919). Later studies of planetary rotations have depended on the fact that a slit placed on the disk parallel to the projection of the rotation axis on the sky should give the same radial velocity at all points along its length. Thus the position angle of the projected axis and the projection of the equatorial velocity on the sky can be determined. In a recent development Vogt and Penrod (1983) have developed a method for analysing distortions of spectrum lines in rotating stars caused by the existence of large star spots to delineate these surface features.

APPLICATIONS TO CLUSTERS

In half a dozen cases notably that of the Hyades, the direction of the proper motions of the member stars show convergence to a point on the sky. The direction to this, by a perspective argument, is the direction of the motion of the whole cluster with respect to the Sun, i.e., each star in the cluster is presumed to be moving in space parallel to this direction with a space velocity which we denote by V_0. The radial component of this, which we can measure is

$$V = V_0 \cos\theta$$

where θ is the angle between the direction to the star and the direction to the convergent point. If the proper motion of this star is μ seconds of arc per year and its parallax is p seconds of arc its transverse velocity is

$$4.74\mu/p = V_0 \sin\theta.$$

From a knowledge of V, θ and μ we can determine V_0 and p. In the case of the Hyades the convergent is at $\alpha = 93°$, $\delta = +12°$, $V_0 = 42\,\mathrm{km\,s^{-1}}$ and the distance (p^{-1} parsecs) is 42 parsecs. The precise values have been the subject of sometimes controversial discussion, since it is not clear that all stars exactly fit the conditions cited above, even if apparently members of the cluster, since although the gravitional binding of members of Galactic clusters is weak, there must be some mutual interaction leading to the establishment of a small velocity dispersion, both in value and direction. The velocity dispersion for a cluster such as the Pleiades is possibly of the order of $0.5\,\mathrm{km\,s^{-1}}$ but is difficult to establish.

The velocity dispersion in globular clusters where gravitional binding is dominant leads to broadening of the spectrum lines since spectra of many stars with different velocities are superposed in such dense fields. Velocity dispersions of the order of $10\,\mathrm{km\,s^{-1}}$ seem to be indicated. Many globular clusters have oblate outlines produced by net rotation of the whole object. 'Equatorial' velocities of the order of 10–$20\,\mathrm{km\,s^{-1}}$ seem to be indicated.

THE VELOCITIES OF FIELD STARS

A number of studies have been made of the velocities of particular types of stars all over the sky. For any given group these may be treated as a sort of cluster and the mean velocity of the group computed in the following manner. Let V be the radial velocity of a member at position (α, δ) then the components of this in equatorial coordinates can be evaluated as $X = V\cos\alpha\cos\delta$, $Y = V\sin\alpha\cos\delta$, $Z = V\sin\delta$ and the mean velocity of n objects computed as

$$X = \textstyle\sum X/n, \quad Y = \sum Y/n, \quad Z = \sum Z/n.$$

This exercise has little realism unless there is some underlying physical unity to the objects observed in each class. This is best brought out by transforming the results to coordinates related to the Galactic system, u, v, w, where these form a *left handed* triad with u directed away from the galactic center, w is to the north side of the Galactic plane and v forms the third axis (actually as we shall see, in the direction of Galactic rotation). The current definition of Galactic coordinates puts the north Galactic pole at $12^{\mathrm{h}}49^{\mathrm{m}}$, $+27°.4$ (1950.0) and the direction to the center is at $17^{\mathrm{h}}42^{\mathrm{m}}.4$, $-28°55'$ (1950.0) (Allen 1973). If the coordinates of the star are for the equinox 1950.0 then these convert to

$$\begin{aligned}
u/V &= 0.0669\cos\alpha\cos\delta + 0.8729\sin\alpha\cos\delta + 0.4836\sin\delta\\
v/V &= 0.4928\cos\alpha\cos\delta - 0.4504\sin\alpha\cos\delta + 0.7445\sin\delta\\
w/V &= -0.8676\cos\alpha\cos\delta - 0.1884\sin\alpha\cos\delta + 0.4602\sin\delta.
\end{aligned}$$

It is impossible to go into great detail concerning the motions of the stars so a brief summary must serve.

Radial velocities alone can be used to define the motion of the Sun with respect to the mean of the group. Somewhat confusingly the mean velocity of a group with respect to the Sun is often called the solar motion of that group. If the group has a degree of homogeneity, e.g., main sequence A-stars of the same apparent magnitude well distributed over the sky, then a meaningful group solar motion can be established and the proper motions, which must exhibit on the average the reflex of this solar motion can be used to determine a statistical parallax for stars in the group and hence their absolute magnitudes.

With respect to the generality of the nearby stars the Sun is usually stated as moving at $19.7\,\mathrm{km\,s^{-1}}$ in the direction $\alpha = 271°$, $\delta = +30°$ (1900) which translates to $u = -10.2$, $v = +15.1$, $w = +7.4\,\mathrm{km\,s^{-1}}$. The motions of most stars are a combination of random and systematic components. The Sun, presumably formed originally as one of a group of objects, possibly now dispersed in space, has a random motion of its own which we see as the solar motion with respect to various groups. Each member of such a group will have its own random or *peculiar motion* and the statistical mean of these *peculiar motions* will measure what is called the velocity dispersion for that group. The solar motion for stars of different spectral classes varies by a few kilometers per second in value and a few degrees in direction. (Allen, 1973).

However, it is more illuminating to note the following. As we shall see the majority of stars in the solar neighborhood are in rotation about the Galactic center following to a first order a circular motion. The velocity of the Sun in this motion is often quoted at $250\,\mathrm{km\,s^{-1}}$ in the direction v though it could be argued that the actual value should be distinctly higher. The components of motion of the Sun with respect to stars thought to be following a strict circular path are often quoted as $u = -9$, $v = +12$, $w = +7\,\mathrm{km\,s^{-1}}$, that is, the solar motion in the sense we use it above should have these signs reversed.

There are types of stars (essentially very old ones, many formed at great distances from the center of the Galaxy) which are in eccentric orbits which carry them mainly in or out of the center which are not wholly participating in the Galactic rotation of the younger solar neighborhood stars. These are thus observed mainly on nearby radial inward or outward tracks and so lag behind the Sun in the Galactic rotation. They are therefore seen as high velocity stars because the Sun is being carried away from them and they therefore have very high solar motions in contrast to the range from 16 to $25\,\mathrm{km\,s^{-1}}$ for the ordinary nearby stars. So called subdwarf stars have solar motions depending on age ranging up to some $250\,\mathrm{km\,s^{-1}}$ and the same is true of RR Lyrae stars as a group. These all belong to an older population often called Halo or Type II stars. (Allen, 1973).

To the extent that stars, all of which orbit the Galactic center, deviate from circular towards elliptical orbits the velocity dispersions of various groups will differ. In general velocity dispersions in the v and w directions are usually about 2/3 of the value for the u direction. For younger stars such as the B-type, this last value is about 10–15 km s^{-1} increasing to about double this for later types. For the Type II stars mentioned above the radial velocity dispersion can be of the order of 130 km s^{-1}. The velocity dispersion in the w direction has been studied as a means of determining the distribution of mass both visible and invisible in the Galactic disk.

In addition to overt clusters of stars which can be recognised by their proximity to each other on the sky, including groups of very early type stars (O and B spectral types) forming ***associations*** many of which appear to be expanding, it has been claimed that embedded in the generality of stars in the sky, are groups of stars, some in widely different directions, which share common motion and are therefore physically related and of common age. A fervent advocate of the existence of these moving groups has been O. J. Eggen who has derived important astrophysical data from their study. (See e.g. Eggen, 1960). The commonality of motion is based on values derived from radial velocities and proper motions and some caution must be exercised since the question arises just how close must the calculated values be to justify ascribing identity to them.

The study of radial velocities has thus been of great importance for undestanding the complexities of the systematic and random motions of individual stars in our Galaxy.

GALACTIC ROTATION

Implicit in our previous remarks has been the idea that at each distance from the Galactic Center there is a certain velocity at which material, whether in the form of stars, or gas, or invisible matter, can move in a circle and that in general it seems likely that the further from the center the smaller this circular velocity will be.

When J. H. Oort studied this topic sixty years ago he envisaged the motion in the Galaxy as a series of separate circles of different radii moving at different circular velocities determined by their radii and the controlling (symmetrically disposed) material interior to each circle. If the observer being carried round by the Sun on some circle observes with a spectroscope stars on circles towards or away from the galactic center ($l = 0°$, or $180°$) he will detect no systematic Doppler shift since all relative motions are transverse not radial. If he observes in directions perpendicular to this ($l = 90°$ or $270°$) he will again see no shift because he is only observing stars moving on the same circle (more fancifully, railroad cars in the same train as himself). However at intermediate angles ($l = 45°$ or $225°$) he will see in the one case a train overtaking him and in the other a train which he is overtaking. At $l = 135°$ and $315°$ what he sees will be objects from which he has a relative velocity of recession.

The upshot of this is that as an observer looks at various places round the Galaxy he should detect a systematic radial velocity due to this differential galactic rotation given by $\rho = rA \sin 2l$, where r is the distance of the objects observed. Clearly this factor enters because the effect will be more pronounced the farther apart the relavant railroad trains are.

The constant A is given by

$$A = \frac{V}{4KR}\left(K - R\frac{dK}{dR}\right)$$

where V is the circular velocity at distance R from the Galactic center and $K(R) = V^2/R$ is, in appropriate units, the acceleration at distance R caused by all the material in the Galaxy. There is a second constant $B = A - V/R$ in principle determinable from proper motion studies.

The value of A is conventionally accepted as 15.0 ± 0.8 km s^{-1} and of B as -10.0 ± 0.8 km s^{-1} Kpc^{-1} with R often quoted at 10 Kpc and V as 250 km s^{-1}.

These values, at least of A, were established as the result of radial velocity measures of B-type stars which have only recently evolved from interstellar gas and reflect what its motion was. The work was undertaken by a number of observatories in the north, notably the Dominion Astrophysical Observatory at Victoria, British Columbia. However, astronomers in the thirties felt very much at a loss having no serious access to the southern sky (they called it "flying on one wing") and thus it was that one of the prime duties laid on the Radcliffe Observatory at Pretoria, South Africa from 1950 onwards was the extension of the work to the southern sky. This was expertly carried out by A. D. Thackeray, A. J. Wesselink and M. W. Feast, who also observed Cepheid variable stars, young evolutionary products of B stars, in the course of extensive researches on Galactic structure. (Feast et al, 1955).

Many of the more distant B-type stars, which usually have broad lines, also show narrow absorption lines, most prominently, the H and K lines of ionised calcium, due to absorption of star light by these atoms on long paths in the Galaxy. Because these lines are formed all along the path the average distance of formation is half that of the star emitting the light, so that differential galactic rotation is detectable, but with a value of A half that for the stars in whose spectra these lines occur.

The reader may detect a certain reserve in the tone of the preceding paragraphs. Although the model of galactic rotation outlined was of great utility and allowed the radio astronomers observing Doppler shifts at 21 cm wavelength emitted by hydrogen to map the distribution of this gas in the Galaxy, it must be acknowledged that the bases of the foregoing discussion are too simple. The general picture of stellar motions is correct, but there are deviations, occasioned in particular by motions associated with the existence of spiral arms and radial motions in our galaxy and continued debate on the exact values of the "constants" involved. Values of A determined as above have so-to-speak mainly local significance, the value of B, which is tied in to the value, V_0 of the local circular velocity, is uncentain and the adopted value of R_0 the distance to the galactic center has fluctuated from 10 kiloparsecs, to 8.2 to 10 and now possibly back to 9 Kpcs over the years. The foregoing discussion is therefore mainly a highly condensed sketch of the history of events in this area of astronomy.

SPECTROSCOPIC BINARY STARS

In 1890 Pickering reported that the spectrum lines of the Dipper star, Mizar, were sometimes double and sometimes single. He correctly attributed this to the fact that the star is actually a pair orbiting round their common center of mass in a period of 20.53860 days. Such stars are known as spectroscopic binaries in which the oribital plane, while not usually passing through the Earth, is not at a large inclination to this, so that the orbital motions show up as regular variations in the Doppler displacements of the lines. The fairly close companion of Mizar (ζ^1 Ursae Majoris) is ζ^2 UMa distant 14 arc seconds on the sky and is itself a spectroscopic binary with a period of 175.55 days. This illustrates a fairly common situation of a quadruple system. Presumably the two pairs ζ^1 and ζ^2 are in motion around their common center of mass, but a rough guess at the period of this motion suggests that it may be more than 1500 years, so that no differential radial velocity is likely to be detectable and no significant positional motion since the invention of the telescope (needed of course to resolve the bright pair).

There are several thousand stars which are known or suspected as showing variable velocity attributable to orbital motion, most of them showing only the spectrum of a single star, the companion being too faint to register, though with the introduction of CCD detectors many previously known as single lined have now had their secondaries detected. Suitably spaced observations of single lined, double lined and even triple lined stars can be used to define a few or many of the orbital parameters of such systems, with, of especial importance, data for the masses of the component stars. Such sources are the principal ones for knowledge of stellar masses not relying on long chains of theoretical argument. The standard catalogue (Batten et al 1978) of orbital elements lists 978 systems for which some or all the physical parameters have been found, but numerous new cases have been added since that publication.

ORBITAL PERIODS

The vast majority of the spectroscopic binaries, for which orbital elements have been derived have periods between 5 and 20 days, though values down to a fraction of a day and up to more than a thousand days are known. The majority are early spectral type stars, B and A, though other types are known, and there may be significant observational selection in favor of these early types. The first parameter to be determined for a spectroscopic binary is its orbital period and though there are now good methods for discovering the values the subject is fraught with pitfalls for those who approach it in a machine like and uncritical way.

In general the data set for a single lined binary will consist of radial velocity values determined on a number of occasions specified by their heliocentric Julian dates, spaced in a somewhat random fashion determined by available telescope time, observing conditions and the skill of choice of times by the observer.

The classical method was to use a trial period, P, guessed in some way, and by computing

$$\text{H.J.D.} \times P^{-1} = \text{integer} + \text{fractional phase}$$

to plot the observed radial velocity against the value of fractional phase (necessarily lying between zero and unity) to see whether the data lay on some kind of plausible curve of velocity against phase. Even though reduced to a computer program for trying many periods as has been done by Deeming (Bopp et al 1970) and Lafler and Kinman (1965), and others, this is still essentially the underlying technique. In an earlier paper Evans (1960) addressed the question, how many trials are needed to determine whether a given range of trial periods does or does not include a possible true answer. It is just as important to exclude some range as to establish that a possible period lies within a given range. The rule given by Evans is that for observations covering N days the spacing in values of P^{-1} should be $0.1/N$ so that for observations covering 100 days $\Delta(P^{-1}) = 0.001$. For example, to test for all periods from 4 to 10 days for a 100 day block of observations takes 150 trials. There is thus a great advantage in having at least one block of observations covering a relatively short period of time from which a rough value of P may be found. Then if there are some outliers covering say 1000 days, trial periods with $\Delta(P^{-1}) = 0.0001$ may be used in the region of the previously suspected value or values. If this procedure is not adopted the work can be enormously extended and even then a false value ('alias') may turn up, simply related to the true value by some such relation as

$$(P)_{\text{true}} = \frac{1}{n}(P)_{\text{false}} \qquad \text{or} \qquad (P^{-1})_{\text{true}} = n \pm (P^{-1})_{\text{false}}$$

where n is a small integer. Even when what seems to be a correct value has been found, it is wise to make further observations, at carefully chosen critical times so that the observed velocity is either maximum positive or minimum negative relative to the mean velocity of the system when the chosen period would lead one to expect these situations. The literature contains some minatory examples of failure to follow such procedures. In particular one should made several observations on the same night to exclude the possibility of very rapid variations. There now exist much more sophisticated techniques for the discovery of periods, sometimes several in one data set, especially from long runs of photometric data in the case of rapid variables such as cataclysmic variables. However for radial velocity work these older, simpler programs are almost always perfectly adequate.

In the case of indistinguishable components of a double-lined binary one should work with absolute values of velocity differences. Each separate value should have a single maximum and a single minimum in the course of a period so that $|V_{\rm I} - V_{\rm II}|$ should have two maxima and two zeros in the course of a period. Then all the relatively positive values in one lobe of the phase diagram belong to star I and so do all the relatively negative values in the other lobe. For a single lined star the velocity is given by

$$V = \gamma + K(e\cos\omega + \cos v + \omega)$$

where γ is the mean velocity, K the half velocity range, e the eccentricity of the orbit, ω the longitude of the ascending node measured from node to periastron and

$$\tan\frac{v}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\frac{E}{2}$$

where $M = 360°(\phi - \phi_0) = E - e \sin E$, and ϕ is the phase computed as above with ϕ_0 the phase of periastron. Once P has been found there are many modern computer programs for evaluating the other parameters.

These are related to the physical parameters of a double lined binary by

$$a_1 \sin i/K_1 = a_2 \sin i/K_2 = 1.375 \times 10^4(1-e^2)^{1/2}$$
$$M_1 \sin^3 i/K_2 = M_2 \sin^3 i/K_1 = 1.039 \times 10^{-7}(K_1+K_2)^2 P(1-e^2)^{3/2}.$$

Here a_1, a_2 are the semi axes majores in kilometers of the orbits of the two components relative to their center of mass, e the eccentricity of these orbits (same for both), P is the period in days, K_1, K_2 the velocity ranges and M_1, M_2 the masses of the components in terms of the sun. The parameter i is the inclination of the orbit which is 90° for an orbit seen edge on. This is not normally determinable from an ordinary double-lined binary, though in the case of eclipsing binaries for which the photometry of the system shows eclipses recurring with the period P, i, which must be large, can be accurately determined from a photometric analysis.

The separate masses cannot be determined for the ordinary double lined spectroscopic binary but their ratio can be, even without a knowledge of P, by plotting V_1, against V_2, which should give a straight line the slope of which gives the ratio, the more mobile star (larger K) being of smaller mass.

For a single-lined binary only the so-called mass function

$$f(M) = M_2^3 \sin^3 i/(M_1+M_2)^2 = 1.0385 \times 10^{-7} K_1^3 P(1-e^2)^{3/2}$$

can be found. In some cases plausible assumptions about the masses M_1 or M_2 can lead to suggestive results. Another situation in which individual masses can be determined obtains when a visual orbit as well as a spectroscopic orbit for a given system is available. In the solution for a visual orbit by itself the data obtained fit the formula

$$(M_1+M_2)Q^2 = A^3 = a^3/p^3$$

where M_1, M_2 are masses in terms of the sun, Q is the period *in years*, A ($= a_1 + a_2$), the semi axis major of the *relative* orbit of M_2 with respect to M_1 in *astronomical units*, a is this value in arc seconds and p the parallax in arc seconds. The observations determine Q, a and the inclination i. The value of p is rarely good enough to sustain being cubed. Even then only the total mass of the system is involved. However if i is known from the visual orbit, and there are spectroscopic observations, M_1 and M_2 can be found separately. A is then known in kilometers and p can be found. This situation until recently occurred rather rarely because sensible variations in radial velocity usually occur only for Q small (in the spectroscopic binary range), a small and p large (restricting choice to a small volume of space).

Recent developments in high resolution astronomy, such as intensity interferometers at Narrabri, Australia and in progress elsewhere, and more importantly speckle interferometry have made it possible to measure visual separations below the classical optical limit of 0.2 are seconds and a number of systems have been investigated so as to yield great expansion of the hitherto severely limited data base for direct determinations of stellar masses. This development has gone along with the production of many dozens of spectroscopic binary orbits by R. F. Griffin (1967) and some others who have followed his lead.

A number of stars in the zodiac have also been resolved by observations of lunar occultations capable of yielding separation data and relative magnitude of components separated by only a few milliseconds of arc.

MULTIPLE STARS

In certain cases spectroscopy has revealed that some star images are in fact the combination of three, four or even more components. Examples are ρ Velorium, β Capricorni, Algol and many more. The spectra of these objects are very complicated and often have a washed outlook because the absorption lines of one or more components fall on the continua of others. Evans (1968) introduced the notion of the mobile diagram based on the model of the type of hanging ornament used by some to decorate their drawing rooms. The whole thing must be suspended above its own mass center. For a typical triple star one object is in orbit with a pair (e.g., β Capricorni) and the pair is orbiting around its own center of gravity. For this latter pair the 'constant' velocity describing its motion is in fact the motion of the center of mass of the pair orbiting the isolated component. To solve the system this secondary motion (variable γ) provides K_2 in the formulae. Then deviations from this 'constant' provide K_1^1 and K_2^1 in the motion of the orbiting pair. The motion about the primary gives $M_1 + M_1^1 + M_2^1$ or the ratio $M_1/(M_1^1 + M_2^1)$. The solution of the secondary pair gives M_1^1 and M_2^1 or M_1^1/M_2^1 or the mass function for this pair. It does not necessarily follow that i for the 'outer' pair is the same as i for the orbiting pair but evidence suggests that it often is approximately so. One complication arises if the 'outer' orbit is large since then the time attributed to each observation may have to be adjusted to allow for the fact that, say, the orbiting pair is sometimes sensibly further away in the line of sight than at other times with the light taking a sensibly longer or shorter time to arrive at the Earth.

In dealing with any multiple star the first desideratum is to construct the mobile diagram and to identify to which component any radial velocity measurement refers. Because with component spectra of any complexity all sorts of accidental blends may show at different velocity complexions of the system, Evans (1968) suggested that in dealing with such multiple spectra the following procedure should be adopted. If lines are split the widest splitting on the spectrum will occur with the closest components in space. If one supposes that a maximum apparent velocity of say, 150 km s^{-1} can occur then if one takes a comparison line, thought to occur in the stellar spectrum and measures every feature within 150 km s^{-1} on either side this ought to include all the real star lines as well as misidentified ones and adventitous blends. Do this for a good many comparison lines and plot a histogram of all the measures. The real values ought to show up against the noise of the misidentifications and other rubbish. This process has indeed been carried out, though with CCD scans a formal procedure of autocorrelation by computer comes to the same thing.

As an example of many of these procedures one may refer to studies of β Capricorni, a triple star where radial velocity data, occultation data and speckle data were all combined to yield masses, colors, spectral types, the system parallax and even the dimensions of the primary. (Evans and Fekel, 1979).

EXTRAGALACTIC STUDIES

The roster of diffuse and extended objects mainly discovered by William and John Herschel was incorporated in Dreyer's New General Catalogue supplemented by his two so-called Index Catalogues which are in fact an indiscriminate mixture of galactic objects such as gaseous and planetary nebulae together with what we now call (external) galaxies. Many of the latter are customarily denoted by their NGC or IC designations. Later the catalogue of Harlow Shapley and Adelaide Ames (1932) segregated the true galaxies in a list of 1300 objects going down to the 13th magnitude. Most recently a comprehensive catalogue of all galaxy data has appeared (Vaucouleurs et al 1976).

Early lists of radial velocities were usually a mixture including predominantly emission objects where the concentration of light into a few lines enabled photographic spectra to be obtained with the slow spectrographs and emulsions then available and were almost all of Galactic objects.

With the improvement of spectrographs and emulsions about 50 years ago systematic measures of galaxy radial velocities, predominantly of recession began to be obtained especially in the USA by such workers as Edwin P. Hubble, Milton Humason and Nicholas U. Mayall leading to the enunciation of Hubble's law of the expansion of the universe in which a general systematic recession of galaxies was established given by

$$\text{recession velocity} = H \times \text{distance.}$$

Round about the time of the second world war the value of H was thought to be about $500\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ based on recession velocities no greater than some ten per cent of the velocity of light.

It was also demonstrated, e.g., by Mayall (1951) in the case of the Andromeda galaxy that in general galaxies exhibited differential rotation much as our own Galaxy had been found to do.

After the war many developments took place. The introduction of image tubes, devices which intensified spectra of objects, reproducing them with enhanced brightness on a screen ready to be photographed enabled the spectra of much fainter galaxies to be studied. Roger Lynds at Kitt Peak was active in this field. Still using ordinary photographic methods but employing a prime focus spectrograph with a long slit Geoffrey and Margaret Burbidge (Burbidge 1961) at McDonald Observatory studied the rotation of many of the more extended galaxies by measuring the deviations of their spectrum lines from perpendicularity caused by Doppler shifts varying with distance from the galaxy center. In the fifties following on Walter Baade's failure to detect RR Lyrae variable stars in the Andromeda galaxy and the success of Thackeray and Wesselink (1953) in detecting them in the Magellanic Clouds it became clear that RR Lyrae stars were not a fainter type of cepheid variable as Shapley had supposed. In fact near the apparent overlap between these two types of object, Cepheids were found to be four times brighter. As Cepheids had been used to estimate distances of the nearby galaxies in which they were detectable as individual stars, this immediately increased estimates of their distances. At the same time studies of the emission objects (H II regions) in galaxies used as a distance indicator for rather more remote ones, suggested that these were much more luminous than had been previously supposed. These considerations suggested that the Hubble constant had been very much overestimated. This overestimate from which the age of the universe since its explosive beginning was deduced, gave an age less than that which was concurrently being derived from age studies based on stellar evolution and terrestrial radioactivity. This led to the proposal of a variety of theories now abandoned that the universe did not have a definite beginning but was continually being renewed by the creation of fresh matter in space. Also about this time G. de Vaucouleurs (1953) from a study of the motions of the nearer galaxies showed that these all belonged to an enormous system — the Local Supergalaxy — which was itself in a rotational motion which affected the recessional motions of the majority of the nearby ones. He contended that a true value of the Hubble constant could only be derived from considering the recession velocities of more remote galaxies where these values would override any local systematic velocities which might be occurring.

To cut a long story short the present value of the Hubble constant has been much reduced, but is still a matter of controversy, the California school led by Allan Sandage supporting a value of near $50\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and the Texas school led by de Vaucouleurs finding a value of $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. The former leads to an age for the Universe of 20 billion years, the latter to 10 billion years.

With the development of radio astronomy following the radar advances of World War II it was found that there existed on the sky large numbers of point sources of radiation. Many of these could be identified with star like faint objects, given the name of 'quasistellar objects', 'quasars' or QSOs. It became possible to obtain optical spectra of some of these which remained mysterious until Marten Schmidt correctly indentified an emission line in the blue region in one case as the ultra violet Lyman alpha line of hydrogen at 1215 A displaced by an enormous Doppler shift. It now became necessary to use the notation z for a Doppler shift explained at the beginning of this contribution. With still further technical developments in which star fields could be studied using television cameras to remove background light and to intensify faint objects, their identification became commonplace, down to magnitudes of 20 or fainter and spectra using charge coupled detector devices obtained. Many quasars show not only a spectrum corresponding to a very large value of z but also superposed, absorption spectra, presumably caused by intergalactic material, corresponding to smaller values. At the time of writing the largest values of z found are about 4.0, corresponding to 92% of the velocity of light and if $H = 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ to a distance of the same proportion of the radius of the universe. Such observations are also looking back in time, for this value to 9.2 billion years ago so that such objects represent an early stage in the evolution of the universe and the formation of galaxies.

Studies on the rotation of galaxies have also be continued intensively, by optical means especially by Vera Rubin (Rubin, 1979) and her associates and have tended to show that typically the circular velocity having reached a certain level continues out unchanged for a very large distance.

At the same time velocities of groups of galaxies have been studied and statistical analyses applied which suggest either that almost all groups are not gravitationally bound or more plausibly that there exist large reservoirs of invisible mass giving such groups a degree of permanence. This is the same conclusion as that drawn from the Rubin studies and has posed one of the capital cosmological problems of the present time, namely that such large invisible masses must exist, though so far their nature is a matter of hot debate.

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Aufgaben zu Abschnitt 7

7.1—7.5 Für die folgenden Stichproben bestimme man die Werte der Verteilungsfunktion und stelle diese Funktion graphisch dar.

7.1 Die Stichprobe in Aufgabe 4.7.

7.2 Würfe mit einem Würfel

6 2 4 1 2 4 3 3 2 1 6 5 6 3 4 1 6 2 5 3

7.3 Die Stichprobe in Aufgabe 4.5.

7.4 Die Stichprobe in Tab. 6.2.

7.5 Die Stichprobe in Aufgabe 6.1.

7.6 Man bestimme und zeichne die Verteilungsfunktion für die Bruttomonatsverdienste männlicher Industriearbeiter in der Bundesrepublik Deutschland im Oktober 1957 (Stat. Jahrbuch f. d. Bundesrep. Deutschland 1960, S. 512).

Verdienst [DM]	0—200	200—400	400—600
Häufigkeit (abgerundet)	2000	88000	358000

Verdienst [DM]	600—800	800—1000	1000—1200
Häufigkeit (abgerundet)	109000	13000	2000

7.7 Man bestimme die Verteilungsfunktion der Stichprobe in Tab. 6.2 auf graphischem Wege unter Benutzung der Abb. 6.1.

7.8 Bei Versuchen zur prophylaktischen Bekämpfung der Nonne (Lymantria monacha L.) wurden an 30 Probestämmen die folgenden Eizahlen beobachtet (C. PALLY, Arch. f. Forstwesen **12**, 1963, 827):

125 212 284 176 100 132 52 319 410 181
273 186 43 11 109 20 76 30 73 47
121 518 129 22 314 144 381 225 257 138

Man überlege sich eine geeignete Klasseneinteilung und stelle die zugehörige Verteilungsfunktion graphisch dar.

Kapitel 3

Mittelwert und Varianz einer Stichprobe

Jede Stichprobe wird durch ihre Häufigkeitsfunktion oder ebensogut durch ihre Summenhäufigkeitsfunktion in allen Einzelheiten beschrieben und bestimmt. Daneben kann man eine Stichprobe auch mehr „summarisch" kennzeichnen, und zwar durch gewisse Werte, die man aus der Häufigkeitsfunktion berechnet und die als **Maßzahlen der Stichprobe** bezeichnet werden. Von diesen Zahlen handelt das vorliegende Kapitel.

8 Mittelwert und Varianz einer Stichprobe

Die beiden praktisch wichtigsten Maßzahlen sind der Mittelwert, der die durchschnittliche Größe der Stichprobenwerte kennzeichnet, und die Varianz, die mißt, wie stark diese Werte streuen.

Der **Mittelwert** einer Stichprobe $x_1, \ldots, x_n$ ist definiert als das arithmetische Mittel der Stichprobenwerte und wird mit $\bar{x}$ bezeichnet. Es ist also

$$\boxed{\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n}.} \tag{8.1}$$

Unter Verwendung des Summenzeichens schreiben wir kürzer

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j .$$

Beispiel 8.1. Fünf Proben von Permalloy-Legierungen mit rechteckiger Hystereseschleife hatten den prozentualen Nickelgehalt

79,4 79,0 78,9 79,2 78,9

(F. PFEIFER, Zeitschr. angew. Physik **13**, 1961, 177). Diese 5 Stichprobenwerte haben die Summe 395,4. Die Stichprobe hat also den Mittelwert

$$\bar{x} = \frac{395{,}4}{5} = 79{,}08\ [\%].$$

Das ist der durchschnittliche prozentuale Nickelgehalt.

Die Stichproben

$$1 \quad 2 \quad 4 \quad 5 \qquad \text{und} \qquad 2{,}7 \quad 3{,}0 \quad 3{,}1 \quad 3{,}2$$

haben beide den Mittelwert 3, unterscheiden sich aber trotzdem recht wesentlich voneinander, denn die Werte der ersten Stichprobe liegen viel weiter auseinander (und auch weiter vom Mittelwert weg) als die Werte der zweiten. Um diesen Unterschied zu erfassen, braucht man noch eine weitere Maßzahl. Geeignet ist hierzu offenbar eine Zahl, die die Abweichung der Stichprobenwerte $x_1, \dots, x_n$ vom Mittelwert $\bar{x}$ mißt. Dabei fordern wir, daß, ähnlich wie beim Mittelwert, *jeder* Stichprobenwert in gewisser Weise mitberücksichtigt wird.

Die wohl am nächsten liegende Möglichkeit, die Summe der Einzelabweichungen $x_1 - \bar{x}, \dots, x_n - \bar{x}$ zu benutzen, scheidet allerdings aus, denn es gilt immer die Beziehung

$$(x_1 - \bar{x}) + (x_2 - \bar{x}) + \cdots + (x_n - \bar{x}) = 0. \tag{8.2}$$

In der Tat steht ja links

$$x_1 + x_2 + \cdots + x_n - n\bar{x},$$

und die Summe der Stichprobenwerte ist gemäß (8.1) gleich $n\bar{x}$.

Unser Fehlschlag kommt daher, daß die Summe in (8.2) im allgemeinen positive und negative Glieder enthält. Er wird vermieden, indem wir entweder zu den Absolutbeträgen oder zu den Quadraten der Glieder $x_j - \bar{x}$ übergehen. Der zweite Weg erfordert zwar etwas mehr Rechenarbeit, erweist sich aber im Hinblick auf die mathematische Begründung der zu entwickelnden statistischen Verfahren als einfacher. Die Maßzahl, die wir auf diesem Wege erhalten, heißt die **Varianz** der Stichprobe $x_1, \dots, x_n$. Sie wird mit s^2 bezeichnet und durch die Formel

$$\boxed{s^2 = \frac{1}{n-1}\sum_{j=1}^{n}(x_j - \bar{x})^2} \qquad (n > 1) \tag{8.3}$$

definiert.

Sind alle x_j zahlenmäßig gleich, so wird $\bar{x} = x_j$, und die Varianz ist gleich null. In jedem anderen Falle gilt

$$s^2 > 0.$$

Die nichtnegative Quadratwurzel der Varianz heißt die **Standardabweichung** und wird mit s bezeichnet.

Daß man gleich beide Größen s^2 *und* s mit einem Namen belegt, kommt daher, daß man beide in der Praxis verwendet. s^2 hat den Vorteil, daß man sich nicht mit Quadratwurzeln herumzuärgern braucht. s hat den Vorteil, daß es dieselbe Dimension (z. B. cm oder kg usw.) wie die x_j besitzt.

Beispiel 8.2. Die beiden obigen Stichproben haben den Umfang $n = 4$, den Mittelwert $\bar{x} = 3$ und die Varianz

$$s^2 = \frac{1}{3}\left[(1-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2\right] \approx 3{,}3$$

bzw.

$$s^2 = \frac{1}{3}\left[(2{,}7-3{,}0)^2 + (3{,}0-3{,}0)^2 + (3{,}1-3{,}0)^2 + (3{,}2-3{,}0)^2\right] \approx 0{,}05.$$

Die Varianz der zweiten Stichprobe ist also wesentlich kleiner als die der ersten, und dasselbe gilt für die Standardabweichungen

$$s \approx \sqrt{3{,}3} \approx 1{,}8 \qquad \text{bzw.} \qquad s \approx \sqrt{0{,}05} \approx 0{,}22.$$

Im Englischen heißt s^2 einheitlich *variance*, und s heißt *standard deviation*. Im Deutschen nennt man s^2 manchmal auch *Streuung*. Leider bezeichnen gewisse Autoren die Standardabweichung s als Streuung. Wenn man solche Literatur liest, muß man also aufpassen.

Man wundert sich vielleicht, daß in (8.3) im Nenner $n - 1$ statt n steht, wie man erwarten sollte. Ein Grund dafür wird in Abschn. 64 angegeben. Man bezeichnet die Zahl $n - 1$ als die Anzahl der *Freiheitsgrade* der Summe der n Quadrate in (8.3).

Aufgaben zu Abschnitt 8

8.1 Man stelle die Häufigkeitsfunktion der Stichproben in Beispiel 8.2 graphisch dar.

8.2–8.4 Man berechne den Mittelwert, die Varianz und die Standardabweichung der folgenden Stichproben.

8.2 1, 2, 3.

8.3 Wirkungsgrad [%] der Kessel einer Hochdruckdampfanlage (R. WLODAWER, Techn. Rundschau Sulzer **43**, 1961, 34)

90,3 91,6 90,9 90,4 90,3 91,0 87,9 89,4.

8.4 Die Stichprobe in Aufgabe 4.4.

8.5 Man konstruiere eine Stichprobe mit dem Umfang 2, dem Mittelwert 0 und der Varianz 1.

8.6 Man beweise: Sind die Werte einer Stichprobe nicht alle zahlenmäßig gleich, so liegt der Mittelwert zwischen dem kleinsten und dem größten Stichprobenwerte.

8.7 Die Differenz zwischen dem größten und dem kleinsten Stichprobenwert heißt die **Spannweite** der betreffenden Stichprobe. Welche Vor- und Nachteile hat es, diese Zahl für die Variabilität der Stichprobenwerte zu benutzen?

8.8 Man berechne die Spannweite der Stichprobe in Aufgabe 8.2.

8.9 Bei gegebenem $x_1, \ldots, x_n$ ist

$$H(t) = \sum_{j=1}^{n} (x_j - t)^2$$

eine Funktion von t. Für welches t wird $H(t)$ am kleinsten?

8.10 Die Werte einer Stichprobe seien in Zentimetern ausgedrückt. Es sei $\bar{x} = 10$, $s^2 = 4$, also $s = 2$. Welche Zahlenwerte ergäben sich für $\bar{x}$, s^2 und s, wenn man die Stichprobenwerte in Millimetern ausdrückte?

9 Vereinfachte Berechnung des Mittelwertes und der Varianz

Wir wollen nun einige Formeln angeben, die für die praktische Berechnung des Mittelwertes und der Varianz meist günstiger als die Definitionsformeln (8.1) und (8.3) sind. Auch der mathematisch weniger Geübte sollte die Mühe nicht scheuen und sich mit den neuen Formeln anfreunden. Es lohnt sich.

Multiplizieren wir jedes Quadrat in (8.3) aus, so besteht jedes Glied aus 3 Summanden. Dann können wir die Summe in (8.3) entsprechend in 3 Summen zerlegen. Ersetzen wir noch $\bar{x}$ gemäß (8.1) durch $\sum x_j/n$, so ergibt sich (vgl. auch Aufgabe 9.1)

$$\boxed{s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2\right].} \tag{9.1}$$

Der Vorteil gegenüber (8.3) ist, daß die Differenzen $x_j - \bar{x}$ nicht auftreten, die kleine Differenzen großer Zahlen sein, also Genauigkeitsverluste bedingen können (die man beim elektronischen Rechnen nicht einmal bemerken würde).

Ersetzen wir $\sum x_j$ in (9.1) gemäß (8.1) durch $n\bar{x}$, so folgt

$$s^2 = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j^2 - n\bar{x}^2\right]. \tag{9.2}$$

Dies ist nicht so gut wie (9.1), weil man beim Bilden von $\bar{x}^2$ durch n^2 dividiert und dann wieder mit n multipliziert.

Aber (9.1) hat für die Tischrechenmaschine oft noch den Nachteil, daß große Zahlen vorkommen. Dem läßt sich durch eine „**Nullpunktverschiebung**“ abhelfen, mit der man zugleich auch $\bar{x}$ einfacher erhält: Wir setzen

$$x_j = c + x_j^* \qquad (\text{also } x_j^* = x_j - c) \tag{9.3}$$

und wählen dabei die Konstante c so, daß die x_j^* kleine handliche Werte haben. Aus der gegebenen Stichprobe $x_1, \ldots, x_n$ erhalten wir dann die „transformierte Stichprobe“ $x_1^*, \ldots, x_n^*$. Wir berechnen nun zuerst den Mittelwert $\bar{x}^*$ der transformierten Stichprobe. Daraus ergibt sich der Mittelwert

$$\bar{x} = c + \bar{x}^* \qquad \left(\text{mit } \bar{x}^* = \frac{1}{n}\sum_{j=1}^{n} x_j^*\right) \tag{9.4}$$

der ursprünglichen Stichprobe. Weiterhin hat diese Stichprobe dieselbe Varianz wie die transformierte. Die einfachen Beweise dieser Tatsachen überlassen wir dem Leser (s. Aufgabe 9.3 und 9.4).

Beispiel 9.1. Für die Stichprobe in Beispiel 8.1 erhalten wir aus (8.3) und Tab. 9.1 die Varianz

$$s^2 = \frac{0{,}1880}{4} = 0{,}047.$$

Aus (9.1) und Tab. 9.2 ergibt sich ebenfalls

$$s^2 = \frac{1}{4}\left[31268{,}42 - \frac{1}{5}(395{,}4)^2\right] = 0{,}047.$$

Viel einfacher wird die Rechnung, wenn wir (9.3) mit $c = 79$ anwenden. Dann ist

$$x_j^* = x_j - 79.$$

Gemäß Tab. 9.3 haben die transformierten Werte den Mittelwert

$$\bar{x}^* = \frac{0{,}4}{5} = 0{,}08.$$

Aus Tab. 9.3 und Formel (9.1) [mit x_j^* statt x_j] ergibt sich die Varianz

$$s^{*2} = \frac{1}{4}\left[0{,}22 - \frac{0{,}4^2}{5}\right] = \frac{0{,}188}{4} = 0{,}047.$$

Wegen (9.4) hat also die gegebene Stichprobe den Mittelwert

$$\bar{x} = 79 + 0{,}08 = 79{,}08$$

und die Varianz $s^2 = 0{,}047$.

Tabelle 9.1–9.3. Zu Beispiel 9.1

Tabelle 9.1

x_j	$x_j - \bar{x}$	$(x_j - \bar{x})^2$
79,4	0,32	0,1024
79,0	−0,08	0,0064
78,9	−0,18	0,0324
79,2	0,12	0.0144
78,9	−0,18	0,0324
	0,00	0,1880

Tabelle 9.2

x_j	x_j^2
79,4	6304,36
79,0	6241,00
78,9	6225,21
79,2	6272,64
78,9	6225,21
395,4	31 268,42

Tabelle 9.3

x_j	x_j^*	x_j^{*2}
79,4	0,4	0,16
79,0	0,0	0,00
78,9	−0,1	0,01
79,2	0,2	0,04
78,9	−0,1	0,01
	0,4	0,22

Die y_i^* hängen mit den y_i durch die Nullpunktsverschiebung (100.3) zusammen. Deshalb ist in (103.5) einfach

$$\bar{y}_i - \eta_i = \bar{y}_i^* - \eta_i^*.$$

So ergibt sich aus Tab. 103.4 der Wert $q_1 = 47{,}2$. Gemäß (103.4) folgt hieraus

$$q_2 = q - q_1 = 112{,}1 - 47{,}2 = 64{,}9$$

und weiterhin aus Tab. 103.5 der Wert $v_0 = 9{,}4/4{,}3 = 2{,}2$.

2. Schritt. Wir wählen die Signifikanzzahl $\alpha = 5\%$.

3. Schritt. Für (5, 15) Freiheitsgrade liefert die Tafel 9a als Lösung der Gleichung

$$P(V \leqq c) = 1 - 0{,}05 = 0{,}95$$

den Wert $c = 2{,}90$. Es ist $v_0 < c$. Wir nehmen deshalb die Hypothese an.

Tabelle 103.4. Zum 1. Schritt in Beispiel 103.2

y_{ij}^*	n_i	$\bar{y}_i^*$	η_i^*	$(\bar{y}_i^* - \eta_i^*)^2$	$n_i(\bar{y}_i^* - \eta_i^*)^2$
1 1	2	1,000	1,535	0,286	0,572
0 2 4	3	2,000	1,688	0,097	0,291
1 1 2 3	4	1,750	1,841	0,008	0,032
0 1 1 6	4	2,000	1,994	0,000	0,000
0 0 2	3	0,667	2,607	3,764	11,292
1 6 7 8	4	5,500	2,913	6,693	26,772
1 2	2	1,500	3,525	4,101	8,202
				Summe	47,161

Tabelle 103.5. Schema der Varianzanalyse in Beispiel 103.2

Variation	Freiheitsgrade	Quadratsumme	Durchschnittsquadrat
Mittelwerte um die Regression	5	47,2	9,4
Innerhalb der Gruppen	15	64,9	4,3
Insgesamt	20	112,1	

Aufgaben zu Abschnitt 103

103.1 Man beweise die Zerlegung (103.4).

103.2 In Beispiel 103.2 berechne man q_2 zur Kontrolle direkt.

103.3—103.5 Unter Benutzung der gegebenen Stichproben teste man jeweils die Linearität der Regression. Dabei nehme man an, daß die in Tab. 103.3 genannte Voraussetzung erfüllt ist.

103.3

x	y	
0	—1	1
1	0	2
2	1	5
4	8	12
6	11	13

103.4

x	y	
1	0	2
2	5	7
3	9	13
4	10	14

103.5

x	y	
2	60	64
5	171	176
10	242	245
15	298	304
20	345	351
25	383	388

In Aufgabe 103.5 ist x die Stromstärke [in mA] und y die Leuchtdichte [in willkürlichen Einheiten] bei einem Kanalstrahlrohr. (R. Gebauer und E. Kreyszig, Zeitschr. f. Physik **135**, 1953, 349—360.)

104 Nichtlineare Regression. Prinzip der kleinsten Quadrate

In Abschn. 94 haben wir gesehen, wie man zu einer Stichprobe $(x_1, y_1), \ldots, (x_n, y_n)$ diejenige Gerade

$$y(x) = bx + k$$

bestimmt, für die die Summe

$$a = \sum_{i=1}^{n} [y_i - y(x_i)]^2 \tag{104.1}$$

der Quadrate der vertikalen Abstände ein Minimum ist. Nach dem entsprechenden Verfahren kann man statt einer Geraden ebensogut eine gekrümmte Kurve, etwa gegeben durch

$$y(x) = b_0 + b_1 x + \cdots + b_m x^m \tag{104.2}$$

bestimmen, wenn dies in einem praktischen Fall angezeigt erscheint. Hierzu berechnet man die n Werte

$$y(x_i) = b_0 + b_1 x_i + \cdots + b_m x_i^m$$

und setzt diese in (104.1) ein. a hat für diejenigen Werte $b_0, b_1, \ldots, b_m$ ein Minimum, für die

$$\frac{\partial a}{\partial b_0} = 0, \qquad \frac{\partial a}{\partial b_1} = 0, \qquad \ldots, \qquad \frac{\partial a}{\partial b_m} = 0 \tag{104.3}$$

ist. Diese $m + 1$ linearen Gleichungen in den $m + 1$ Unbekannten $b_0, \ldots, b_m$ heißen die **Normalgleichungen.** Ihre Lösung gibt die gesuchten Werte der Koeffizienten in (104.2). Theoretische Schwierigkeiten bestehen nicht. Praktisch kann das Lösen dieser Gleichungen natürlich oftmals noch erhebliche Rechenarbeit verursachen. Ein wichtiges Verfahren zur Lösung eines solchen Gleichungssystems ist der sogenannte **Gaußsche Algorithmus,** ein systematisches Eliminationsverfahren, das wir nachstehend an einem Beispiel (Beispiel 104.1) erläutern.

Im Falle einer Parabel

$$y(x) = b_0 + b_1 x + b_2 x^2 \tag{104.4}$$

ist

$$y(x_i) = b_0 + b_1 x_i + b_2 x_i^2.$$

Dann gewinnt (104.1) die Gestalt

$$a = \sum_{i=1}^{n} (y_i - b_0 - b_1 x_i - b_2 x_i^2)^2.$$

Hieraus ergibt sich durch Differentiation

$$\frac{\partial a}{\partial b_0} = -2 \sum (y_i - b_0 - b_1 x_i - b_2 x_i^2) = 0$$

$$\frac{\partial a}{\partial b_1} = -2 \sum x_i (y_i - b_0 - b_1 x_i - b_2 x_i^2) = 0$$

$$\frac{\partial a}{\partial b_2} = -2 \sum x_i^2 (y_i - b_0 - b_1 x_i - b_2 x_i^2) = 0.$$

Das sind 3 lineare Gleichungen mit den 3 Unbekannten b_0, b_1 und b_2. Diese Normalgleichungen können wir in der folgenden Form schreiben:

$$\begin{aligned} b_0 n + b_1 \sum x_i + b_2 \sum x_i^2 &= \sum y_i \\ b_0 \sum x_i + b_1 \sum x_i^2 + b_2 \sum x_i^3 &= \sum x_i y_i \\ b_0 \sum x_i^2 + b_1 \sum x_i^3 + b_2 \sum x_i^4 &= \sum x_i^2 y_i \end{aligned} \tag{104.5}$$

Tabelle 104.1 Prozentuale Sterblichkeit von Neugeborenen (Schwangerschaftsdauer 280—289 Tage post menstr.) in Abhängigkeit von der Körperlänge bei der Geburt (H. HOSEMANN, Die Naturwiss. **37**, 1950, 410)

Körperlänge		Anzahl Neugeborener in der Klasse	Davon gestorben	Sterblichkeit y [%]
Klassenintervall	Klassenmittelpunkt x_i [cm]			
46—49	47,5	124	14	11,29
49—52	50,5	1255	13	1,04
52—55	53,5	3149	28	0,89
55—58	56,5	1441	31	2,15
58—61	59,5	175	16	9,14

Beispiel 104.1 (Sterblichkeit Neugeborener in Abhängigkeit von der Körperlänge). Stellt man die Werte der gruppierten Stichprobe in Tab. 104.1 graphisch dar, so erhält man die Abb. 104.1. Daraus sieht man, daß es naheliegt, die Regression der Sterblichkeit y bezüglich der Körperlänge x durch eine Parabel zu beschreiben. Die Koeffizienten dieser Parabel (104.4) erhalten wir durch Aufstellen und Lösen des Gleichungssystems (104.5).

Tabelle 104.2. Zu Beispiel 104.1

$x_i^* = x_i - 53{,}5$	x_i^{*2}	x_i^{*3}	x_i^{*4}	Anzahl	y_i	$x_i^* y_i$	$x_i^{*2} y_i$
−6	36	−216	1296	124	11,29	−67,74	406,44
−3	9	− 27	81	1255	1,04	− 3,12	9,36
0	0	0	0	3149	0,89	0	0
3	9	27	81	1441	2,15	6,45	19,35
6	36	216	1296	175	9,14	54,84	329,04

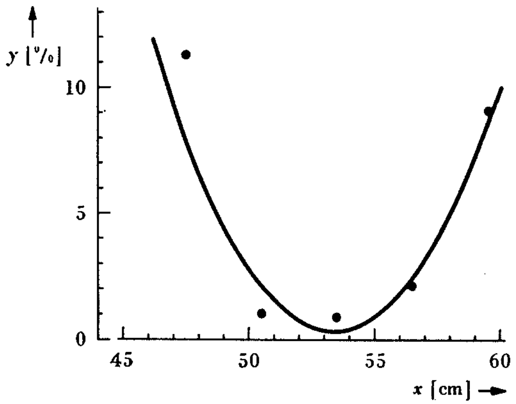

Abbildung 104.1. Die Stichprobe in Tab. 104.1 und die Parabel (104.9)

Die Rechnung wird bequemer, indem wir

$$x_i = x_i^* + 53{,}5 \qquad (\text{also } x_i^* = x_i - 53{,}5)$$

setzen. Wie wir wissen, denkt man sich bei einer in Klassen eingeteilten Stichprobe die Stichprobenwerte jeweils im Klassenmittelpunkte liegend. Aus Tab. 104.2 erhalten wir demnach

$$\begin{aligned} \sum x_i^* &= 124 \cdot (-6) + 1255 \cdot (-3) + 1441 \cdot 3 + 175 \cdot 6 = 864 \\ \sum x_i^{*2} &= 124 \cdot 36 + 1255 \cdot 9 + 1441 \cdot 9 + 175 \cdot 36 = 35028 \\ \sum x_i^{*3} &= 124 \cdot (-216) + 1255 \cdot (-27) + 1441 \cdot 27 + 175 \cdot 216 = 16038 \\ \sum x_i^{*4} &= 124 \cdot 1296 + 1255 \cdot 81 + 1441 \cdot 81 + 175 \cdot 1296 = 605880 \end{aligned}$$

denn die Stichprobe besteht aus $n = 6144$ Wertepaaren, in denen man sich nach der Klassenbildung $x_1^* = -6$ genau 124mal, $x_2^* = -3$ genau 1255mal, $\ldots$, $x_5^* = 6$ genau 175mal vorkommend denkt. Entsprechend wird

$$\begin{aligned} \sum y_i &= 124 \cdot 11{,}29 + 1255 \cdot 1{,}04 + \cdots = 10205{,}42 \\ \sum x_i^* y_i &= 124 \cdot (-6) \cdot 11{,}29 + 1255 \cdot (-3) \cdot 1{,}04 + \cdots = 6576{,}09 \\ \sum x_i^{*2} y_i &= 124 \cdot 36 \cdot 11{,}29 + 1255 \cdot 9 \cdot 1{,}04 + \cdots = 147610{,}71. \end{aligned}$$